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RITHMOMACHIA, THE GREAT MEDIEVAL NUMBER GAME.

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When the subject of number games shall be adequately treated, and the long and interesting story is told of how the world has learned to handle the smaller numbers quite as much through play as through commerce, the climax will probably be found in the chapter relating to the Battle of Numbers, the Rithmomachia of the Middle Ages. For here was a tournament worthy of intellectual foes, a play that outranked chess as much as chess surpasses mere dicing, and a game that was by its very nature closed to all save selected minds that had been trained in the Boethian arithmetic, the Latinized Nicomachus, the last great effort in the Pythagorean philosophy of numbers.

But when this story comes to be told the one who relates it will have no easy task, and the object of this paper is rather to set forth the problem than to solve it. For although we have manuscripts of three writers of the eleventh century, two of the twelfth, one of the thirteenth, and Bradwardin's work of the fourteenth,* and although we have several printed treatises on the subject,† we know practically nothing of the origin of the game. We only know that the medieval writers attributed it to Pythagoras, that no trace of it has been discovered in Greek literature, and that no mention of it has been found before the time of Hermannus Contractus (1013-1054). The name, which appears in a variety of forms,‡ points to a Greek origin, the more so because Greek was little known at the time when the game first appears in literature. Based as it is upon the Greek theory of numbers,

* There is a twelfth century manuscript of Hermannus Contractus (1013-1054) at Paris, and others of later date in various libraries. Wappler has published this, and also a treatise by Asilo (before 1077), with part of an anonymous one of the twelfth century. Odo also wrote on the subject in the eleventh century. Peiper has edited Fortolfus's work of the twelfth century. Several other manuscripts are known. Consult Wappler, in the *Zeitschrift für Mathematik und Physik*, Vol. 37, p. 1 (1892), and Peiper in the *Abhandlungen*, Vol. 3 (1880).

† We have made free use of the brief description given by Jacobus Faber Stapulensis (1496), possibly from Shirewood's manuscript, and the works of Boissière (French edition 1554, Latin edition 1556) and Barozzi (1572). Abraham Riese (1562) published Asilo's manuscript.

‡ Correctly, Rithmomachia, but also in such incorrect forms as Rythmomachia (Battle of Rhythms), Rithmimachia, Rythmimachia, etc.

appearing as it does with a Greek name, necessarily a game known to but few and one that would naturally pass from the élite to the élite, attracting no attention from the populace, it is easy to feel that the origin of the game is to be sought in the Greek civilization, and perhaps in the later schools of Byzantium or Alexandria. The very fact that a game so well known as to justify printed treatises in Latin, French, Italian, and German, in the sixteenth century, and to have public advertisements of the sale of the board and pieces under the shadow of the old Sorbonne,* and that this game has been forgotten for upwards of three centuries of modern civilization, shows how easily it might have left no record during the period which we so truly designate as the dark ages. Furthermore the early manuscripts are so obscure and condensed as to show that they presupposed a knowledge of the game, merely summarizing some of the more difficult rules to be followed, so that it would seem a proper conjecture that scholars were transmitting it by word of mouth, only recording now and then a few directions that were not so easily retained in the memory.

The game was played on a board resembling the one used for chess or checkers, with eight squares on the shorter side, but with sixteen on the longer side. The forms used for the pieces were triangles, squares, circles, and pyramids, and the pieces were set as is shown in the illustration (Fig. 1) here given from Boissière. In this setting the white pieces are numbered in the same way as the black pieces in the work by Jacobus Faber Stapulensis, so that the color had no significance. The names of the pieces and their position before the opening of the game is shown in Fig. 2, the lower ones being called the Evens (starting from the even numbers 2, 4, 6, 8), and the upper ones being called the Odds. The dots placed below the numbers serve to mark the bottom of the piece, so that 6 shall not be confused with 9, 81 with 18, etc.

The first row on each side (Fig. 2) is made of the odd and even numbers, respectively, unity being admitted as not a number "*sed fons et origo numerorum.*" The second row is made up of the squares of the first. The sum of the two rows gives the first row of triangles. The second row of triangles is formed from the first one by means of a relation known as superparticularis; that is, a number in the second row is found by joining the corresponding number in the first row to an aliquot part of it determined by the number in the first circle at the top. For example, 81 is obtained from 72 by adding $\frac{1}{8}$ of 72, 8 being the number in the circle at the top of that column. Similarly, $42=36+\frac{1}{6}$ of 36, $20=16+\frac{1}{4}$ of 16, and $6=4+\frac{1}{2}$ of 4. The ratio of the aliquot parts added, to the numbers at the top, are therefore $\frac{3}{2}$ (sesquialtera), $\frac{5}{4}$ (sesquiquarta), etc. The relation of the lower triangles of each side to the lower circles of the other side does not seem to have been noticed. The first row of squares is formed by adding the respective triangles ($9+6=15$, $25+20=45$, etc.), but one square on

* One of the editions of Boissière advertises the sale of this material at the shop of "John the Gentile."

each side (91 for the Evens, and 190 for the Odds) is replaced by a pyramid. These pyramids are formed by superposing squares (thin prisms), and are used to call attention to the peculiar construction of the numbers which they represent. Thus $91=6^2+5^2+4^2+3^2+2^2+1$, and $190=8^2+7^2+6^2+5^2+4^2$. Since the former contained the squares of all numbers from 1 to 6, it was called a perfect pyramid, but since the latter lacked $3^2+2^2+1^2$ it was known as *tricurta* (thrice curtailed),

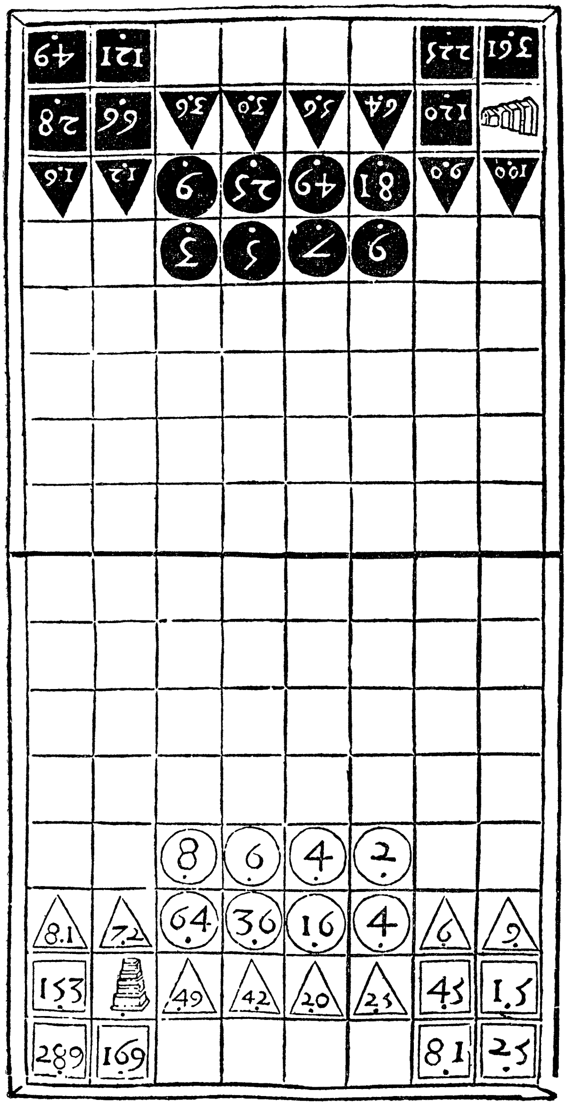


Fig. 1. From Boissière's work of 1554, 1556.

The lower row of squares is obtained from the upper one by a formula somewhat like the one used in obtaining the lower row of triangles. In the case of the squares, if we call the number in the circle at the top n, and the number in the upper square s, then the number in the lower square is $\left(\frac{2n+1}{n+1}\right)s$. Thus $\left(\frac{2 \cdot 2+1}{2+1}\right) \cdot 15 = \frac{5}{3} \cdot 15 = 25$; $\left(\frac{2 \cdot 4+1}{4+1}\right) \cdot 45 = \frac{9}{5} \cdot 45 = 81$, and so on. The ratio between 25 and 15, $\frac{5}{3}$, is one of the superpartientes, namely, the superbipartientes (surpassing by two parts). It should be said, however, that other rules are given for the derivation of these numbers, and that there are slight variations in the pieces, but these have no significance.

The pieces are now arranged for the opening of the play in the manner shown in Fig. 1. The players move the pieces in turn, as in chess. A circle moves one space, a triangle three, and a square four. The game consists in capturing an opponent's pieces, this being effected in one of four ways—by meeting, by assault, by ambuscade, and by siege.

The method by meeting may be illustrated as follows: If Even's triangle 25 can, by advancing three spaces, reach Odd's circle 25, Even does not move his piece, but simply takes up his opponent's.

The capture by assault is affected in this way: If a smaller number, multiplied by the number of vacant spaces between it and a larger one equals the larger one, it may take it. For example, Odd's circle 5 may take Even's square 45 if nine separates the two. This requires the players to be familiar with the multiplication table, and for this purpose Fortolfus provides the usual square array known in the Middle Ages as the *mensa Pythagorica.*

The capture by ambuscade is as follows: If two pieces whose sum equals the number on an opponent's piece can be moved into the spaces on either side of it, the latter is ambuscaded and must surrender. For example, to capture Odd's triangle 12, Even's circles 4 and 8 must be able to move on either side of it.

The capture by siege is effected if a piece is immediately surrounded on all four sides by opposing pieces; that is, if the adjacent spaces above, below, to the right, and to the left are filled.

It is evident that a pyramid can rarely be taken except by siege. Interest was therefore added to the game by making it subject to several attacks. A pyramid was considered to be in danger whenever one of its laminæ was attacked by any one of the four methods. In this case a ransom was allowable, namely, a piece of the same value as the base. In case no such piece could be offered because of prior capture, any other piece could be given that the opponent might be willing to accept. Positive capture of the piece not being possible if the numbers 91 and 190 were retained, it was permitted if the base square was successfully attacked, namely, 36 or 64. The piece having no particular value, its loss was no more serious than that of any other piece, but the plans of attack were more interesting.

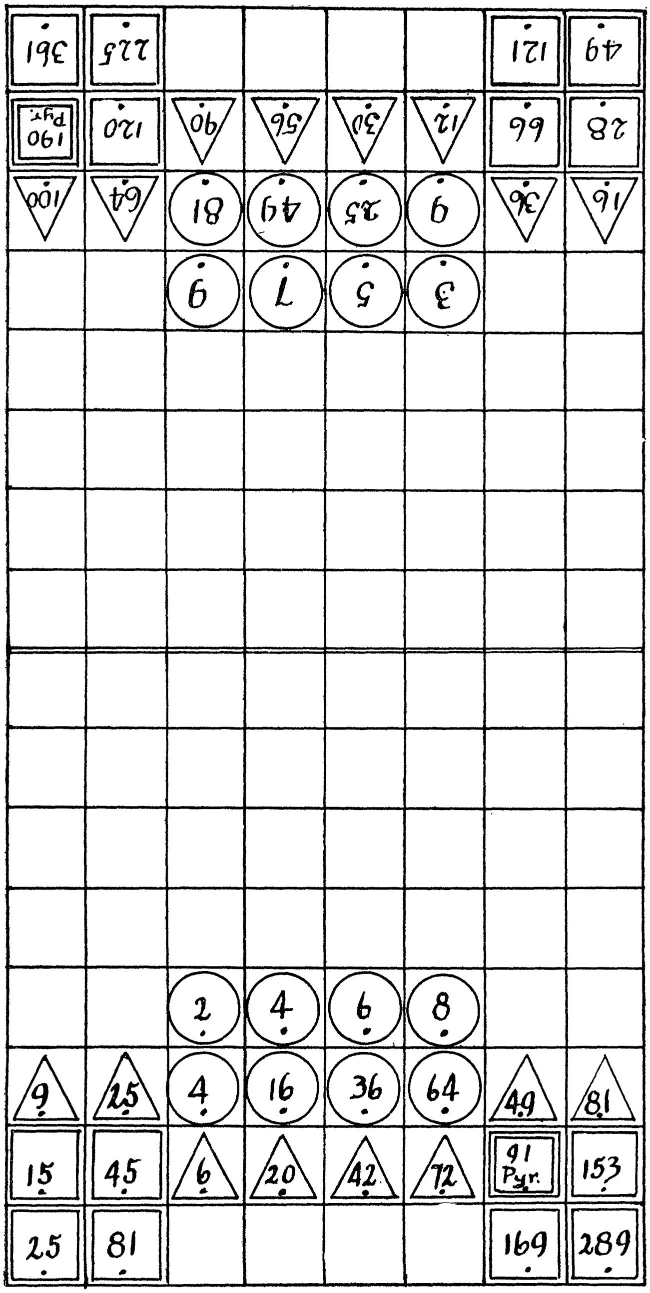


Fig. 2. The Setting of the Pieces.

As already stated, the game consists in capturing an opponent's pieces. This, however, is not all there is of it. The capture is undertaken for the purpose of obtaining what is technically called a Victory, and the rules provide for no less than eight of these Victories. Before beginning to play, the particular kind of Victory for which the contest is to be waged is agreed upon by the parties. Five of these kinds are known as Common Victories, and the rest as Proper Victories, the former being considered as suited to tyros and the latter as worthy of veteran players.

Common Victories were, as already said, of five kinds, as follows: (1) Victory *de corpore*, decided by the number of pieces captured; (2) Victory *de bonis*, depending upon the value of the pieces; (3) Victory *de lite*, depending not only upon the value of the pieces but upon the number of the digits inscribed upon them; (4) Victory *de honore*, depending upon both the number of the pieces and their value; (5) Victory *de honore liteque*, depending upon the number of pieces, their value, and the number of digits inscribed upon them.

If, for example, the players decided upon the Victory *de corpore*, they would agree in advance upon some number, as twenty-four, as the winning number. As soon as either player captured twenty-four of the opponent's pieces he won the game.

If they decided in advance upon the Victory *de bonis*, they would agree upon some number like 160 as the winning number. Each player would then seek to capture pieces of which the sum of the values should equal or exceed 160.

If they decided upon the Victory *de lite* they might again select 160 with the further condition that the total number of digits on the pieces should equal some small number such as eight. A player would then try to capture pieces like 56, 64, 28, and 15, but would not try for 121, 9, and 30.

If the victory was to be *de honore* the players would agree upon some number like 160 for the sum of the values, and some other number like five for the number of the pieces. In this case neither 56, 64, 28, 12 nor 121, 9, 30 would suffice, but 64, 36, 30, 25, 5 would meet the two conditions.

In the Victory *de honore liteque* the players might agree upon 160 for the values, five for the number of pieces, and nine for the number of digits. These conditions are satisfied by 64, 36, 30, 25, 5 from the Odd's pieces, and 64, 36, 42, 16, 2 from the Even's pieces.

It is already evident that the game has more of merit than at first seemed probable. These Common Victories do not, however, show it as played by the real lover of Rithmomachia, It was in the Proper Victories that he found a game worthy of his efforts, and with these we shall close this description.

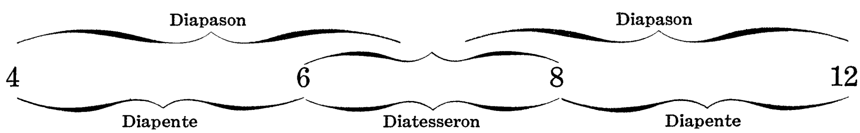
The Proper Victories were known by the names of Magna, Major, and Praestantissima, and they resulted from combinations relating to the three best known types of progressions, the arithmetic, geometric, and harmonic—

progressions that had come down through the Greek mathematics from the Pythagoreans. In each of these victories the pieces, one of which must be taken from the opposing side, must be displayed in the selected progression from the opponent's side of the board.

The Victoria Magna consists in arranging three counters in any one of the three simple progressions. There are forty-one combinations that make possible such an arrangement in arithmetic progression, eighteen in geometric progression, and seventeen in harmonic progression. The possibilities are greater for Even in the first case and for Odd in the second case, and they are equal in the third case. One of these arrangements, in geometric progression, is 6, 8, 12. To this Fortolfus gave the name Cubic Victory, the first number representing the faces, the second the vertices, and the third the edges of a cube.

The earlier writers gave to the second of the Proper Victories the name Victoria Minor, but Boissière calls it Victoria Major. In this there are combined two progressions, arithmetic and geometric, geometric and harmonic, or harmonic and arithmetic. To secure this victory four pieces must be brought in line in the enemy's field, two of which must belong to one of the selected progressions and two to the other. For example, 2, 3, 4, 8 would gain a Victoria Major for either Even or Odd, for 2, 3, 4 are in arithmetic progression and 2, 4, 8 are in geometric, where 2, 4, 8 are Even's pieces and 3 is Odd's piece. There are in all sixty-one such double progressions, all but one of which can be used by Even, and all of which can be used by Odd.

The climax of the game was reached in the Victoria Praestantissima, or Victoria Excellentissima. In this victory it was necessary to get four numbers in a row, which numbers embodied all three progressions. There are only six possible solutions to this problem, namely, (2, 3, 4, 6), (4, 6, 8, 12), (7, 8, 9, 12), (4, 6, 9, 12), (3, 5, 15, 25), (12, 15, 16, 20). Of these only one (4, 6, 9, 12) contains four terms in geometric progression, the others containing proportions. Upon these combinations the early writers dilate with not a little affection. For example, in the set 4, 6, 8, 12, the comparison of 12 and 8, or of 6 and 4, is a sesquialtera ($\frac{3}{2}$), corresponding to the fifth in music; 8 and 4, or 12 and 6, have the ratio 2:1, that of the octave or diapason. The ratio 8:6 gives the diatesseron, while 12:4 gives the interval, a twelfth, including both diapason and diapente. The ratio 8:(6—4) gives the interval of two octaves, or a fifteenth, and all of this was shown in graphic form as follows:



Such is a brief description of the game of which Boissière speaks as "*Noblissimus et antiquissimus ludus Pythagoreus qui Rythmomachia nominatur,*"* a game of which we are told many of the devotees were men of no mean reputation. Such leaders of thought as Gerbert, whom his contemporaries called a wizard but made a Pope; Hermannus, whose infirmity gave him the name of Contractus, by which he is commonly known; Robertus Castrensis, who helped to make the Arab learning known;† Nicolaus Horem, who became Bishop of Lisieux in 1377, and who sought to enrich the intellectual world by his teaching of the ancient theory of numbers; Oronce Finé (Orontius Finaeus), who was professor of mathematics in the (later called) Collége de France in 1532; Jacques le Fèvre d'Estaples (Jacobus Faber Stapulensis), the learned tutor of the son of François I; Thomas Bradwardin, who died in 1349 as Archbishop of Canterbury, and who, from his great learning, was known as "Doctor Profundus;" John Shirwood (Shirewode), who died in 1494 as Bishop of Durham—these are the names of some of those who played the game, and several of them composed tractatês setting forth its merits. It cannot be revived, since the interest in the number theory for which it stood has passed away, but even some slight understanding of its nature cannot fail to have interest for any one who takes pleasure in mathematics, in education, or in the evolution of both mathematics and education from the ideals of the Greek philosophy to the ideals of the present day.

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* For the full title in facsimile, see Smith, *Rara Arithmetica,* Boston, 1909, p. 272. with other references in the index.

† Professor Karpinski of the University of Michigan, is now working on one of his translations, the algebra of Al-Khowarazmi.